A Multiplicative Spectral Characterization of Characters of $C^\star$-algebras

R. Brits, M. Mabrouk, F. Schulz, G. Sebastian and C. Touré

Banach Algebras and their Applications

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- If χ is a character of A then for each $x \in A$, $\chi(x) \in \sigma(x)$

Motivation

Theorem 1 (Gleason-Kahane-Żelazko, 1967-1968)

Let A be a complex Banach algebra. Then a linear functional $\phi : A \to \mathbb{C}$ is a character of A if and only if $\phi(x) \in \sigma(x)$ for each $x \in A$.

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Theorem 2 (Kowalski-Słodkowski, 1980)

Let A be a complex Banach algebra. Then a functional $\phi : A \to \mathbb{C}$ is a character of A if and only if

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So, this beckons the question: Are there multiplicative versions of these results? That is, if we replace "linear" by "multiplicative" in Gleason-Kahane-Żelazko and $+$ by $\times$ in Kowalski-Słodkowski, are the results still valid?

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- Consider the following: Let $A = \mathbb{C}^2$ with the usual pointwise operations and any algebra norm. Define $\phi : A \to \mathbb{C}$ by

$$\phi((\alpha, \gamma)) = \begin{cases} \gamma & \alpha \neq 0 \\ 0 & \alpha = 0. \end{cases}$$

  It is easy to see that ϕ is not linear but that it is multiplicative (and also homogeneous). Moreover, $\phi(x) \in \sigma(x)$ for each $x \in A$.

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  It is easy to see that ϕ is not linear but that it is multiplicative (and also homogeneous). Moreover, $\phi(x) \in \sigma(x)$ for each $x \in A$.
- However, ϕ is clearly not continuous.
- It seems reasonable to add continuity to the assumptions in the multiplicative problem.

Further motivation comes from an old result of Carleson (of the Corona Problem):

Theorem 3 (Carleson, 1957)

Let ϕ be a nonzero continuous multiplicative functional on A, and let $x \in A$ be arbitrary. Then the map

$$\lambda \mapsto \log |\phi(x - \lambda \mathbf{1})|$$

is harmonic on the unbounded connected component of $\mathbb{C} \setminus \sigma(x)$.

So continuous multiplicative functionals do exhibit some good behaviour...

Earlier results

Theorem 4 (Maouche, 1996)

Let A be a Banach algebra, and let $\phi : A \to \mathbb{C}$ be a multiplicative function satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$. Then, corresponding to ϕ, there exists a unique character on A which agrees with ϕ on $G_{\mathbf{1}}(A)$.

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Theorem 5 (Brits, Schulz, Touré, 2017)

Let A be a Banach algebra, and let ϕ be a multiplicative functional on A satisfying $\phi(x) \in \sigma(x)$ for each $x \in A$. Then the following are equivalent:

- *ϕ is a character of A.*

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- *For each $x \in A$ the map $\lambda \mapsto \phi(\lambda\mathbf{1} - x)$ is an entire function on $\mathbb{C}$.*
- *For each $x \in A$ the map $\lambda \mapsto |\phi(x - \lambda\mathbf{1}) + \lambda|$ is subharmonic on $\mathbb{C}$.*

Theorem 6 (Brits, Schulz, Touré, 2017)

Let A be a von Neumann algebra and let $\phi : A \to \mathbb{C}$ be a multiplicative functional. Then ϕ is a character of A if and only if ϕ is continuous and $\phi(x) \in \sigma(x)$ for each $x \in A$.

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Theorem 7 (Brits, Mabrouk, Touré, 2021)

Let A be any $C^\star$-algebra and let $\phi : A \to \mathbb{C}$ be a multiplicative functional. Then ϕ is a character of A if and only if ϕ is continuous and $\phi(x) \in \sigma(x)$ for each $x \in A$.

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Theorem 8 (Brits, Schulz, Touré, 2018)

Let A be a Banach algebra such that $\sigma(x)$ is totally disconnected for each $x \in A$. If a functional ϕ on A is continuous and satisfies $\phi(x)\phi(y) \in \sigma(xy)$ for all $x, y \in A$, then either ϕ or $-\phi$ is a character of A. In particular if A is a scattered Banach algebra then ϕ or $-\phi$ is a character of A

Spectrally Multiplicative Functionals on $C^\star$-Algebras

Throughout this section A is a unital $C^\star$-algebra. We denote by $\mathcal{S}$ the collection of all self-adjoint elements of A. We shall consider a function $\phi : A \to \mathbb{C}$ satisfying the following conditions:

(P1) $\phi(x)\phi(y) \in \sigma(xy)$ for all $x, y \in A$,

(P2) $\phi(\mathbf{1}) = 1$,

(P3) ϕ is continuous on A.

and refer to a functional satisfying (P1)-(P2) as a spectrally multiplicative functional.

Lemma 9

Let $x \in \mathcal{S}$. *If* $\phi(x) \neq 0$, *then:*

(i) $\phi(\mathbf{1} + ix) = 1 + i\phi(x)$,

(ii) $\phi(tx) = t\phi(x)$, *for each* $t \in \mathbb{R}$,

(iii) $\phi(e^{tx}) = e^{\phi(tx)} = e^{t\phi(x)}$, *for each* $t \in \mathbb{R}$,

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$$\phi(e^x) + i\phi(e^x)\phi(x) = e^{\gamma} + e^{\gamma}\gamma i$$

for some $\gamma \in \sigma(x)$.

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for some $\gamma \in \sigma(x)$.
- Consequently $\phi(e^x) = e^\gamma$ and $\phi(x) = \gamma$ and so $\phi(e^x) = e^{\phi(x)}$.

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ϕ has the following properties:

(i) *If $x \in \mathcal{S}$, then $\phi\left(e^{\lambda x}\right) = e^{\lambda \phi(x)}$ holds for all $\lambda \in \mathbb{C}$.*

(ii) *If $x, x_1, \ldots, x_n \in \mathcal{S}$, and $\lambda \in \mathbb{C}$, then*

$$\phi\left(e^{\lambda x} e^{x_1} \cdots e^{x_n}\right) = \phi\left(e^{\lambda x}\right) \phi\left(e^{x_1}\right) \cdots \phi\left(e^{x_n}\right).$$

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- Since ϕ takes real values on $\mathcal{S}$ we have the result.

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The formula

$$\psi_\phi(x) := \phi\left(\mathrm{Re}(x)\right) + i\phi\left(\mathrm{Im}(x)\right)$$

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$$\left[\phi\left(e^{tu}\right)\phi\left(e^{itv}\right) - 1\right]/t \in \sigma\left(\left[e^{tu}e^{itv} - \mathbf{1}\right]/t\right).$$

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- If we let $t \to 0$, then, using the fact that $A \setminus G(A)$ is closed in A, it follows that $\psi_\phi(x) = \phi(u) + i\phi(v) \in \sigma(u + i\,v)$.

Lemma 12

Let x be an element of $\mathcal{S}$. Then

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Proof

- We shall prove the result where a is any positive element of A: We can assume without loss of generality that A is commutative so that $A = C(X)$ for some compact set X. Define $b_n = ae^{-na}$ and $c_n = a\left(\mathbf{1} + ina\right)^{-1}$.

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$$\|b_n\| = \sup\left\{a(x)e^{-na(x)} : x \in X\right\} \leq \sup\left\{te^{-nt} : t \geq 0\right\} \leq \frac{e^{-1}}{n}$$

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- From (P1) it follows that

$$\phi(x) = \phi(x)\phi(W_n) \in \sigma(xW_n) = \sigma(uW_n + ivW_n). \tag{1}$$

Lemma 13

Let ϕ be a continuous spectrally multiplicative functional on a $C^\star$-algebra A, and suppose $x \in A$ satisfies $\psi_\phi(x) = 0$. Then $\phi(x) = 0$.

Proof

- Write $x = u + iv$ and assume $\psi_\phi(x) = 0$.
- For each $n \in \mathbb{N}$ let $W_n := e^{-n\sqrt{|u|^2+|v|^2}}$ and observe that $\phi\left(W_n\right) = \psi_\phi\left(W_n\right) = 1$.
- From (P1) it follows that

$$\phi(x) = \phi(x)\phi\left(W_n\right) \in \sigma\left(xW_n\right) = \sigma\left(uW_n + ivW_n\right). \qquad (1)$$

- Using Lemma 12 we deduce that

$$\lim_n \sqrt{|u|^2 + |v|^2}\, W_n = \mathbf{0} \implies \lim_n |u| W_n = \mathbf{0} \implies \lim_n uW_n = \mathbf{0},$$

  and similarly $\lim_n vW_n = \mathbf{0}$. Thus

$$\lim_n xW_n = \lim_n \left(uW_n + ivW_n\right) = \mathbf{0}.$$

  and the result follows.

Lemma 14

Let ϕ be a continuous spectrally multiplicative functional on a $C^\star$-algebra A, $\alpha \in \mathbb{C}$, and suppose $x \in A$ satisfies $\psi_\phi(x) = 0$. Then $\phi(\alpha\mathbf{1} + x) = c_\alpha \alpha$, for some $c_\alpha \in [0, 1]$.

Lemma 14

Let ϕ be a continuous spectrally multiplicative functional on a $C^\star$-algebra A, $\alpha \in \mathbb{C}$, and suppose $x \in A$ satisfies $\psi_\phi(x) = 0$. Then $\phi(\alpha \mathbf{1} + x) = c_\alpha \alpha$, for some $c_\alpha \in [0, 1]$.

Proof

- With $W_n := e^{-n\sqrt{|u|^2+|v|^2}}$ let $Y_n := \frac{1}{\alpha} x W_n$, and set $c_\alpha := \frac{1}{\alpha}\phi(\alpha \mathbf{1} + x)$. From earlier we have $\lim_n Y_n = \mathbf{0}$.

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- Then

$$c_\alpha = \frac{1}{\alpha}\phi(\alpha\mathbf{1} + x)\phi\left(W_n\right) \in \sigma\left(W_n + Y_n\right). \tag{2}$$

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- Assume, to the contrary, that $c_\alpha \notin [0,1]$. For each n, we have that $W_n \in \mathcal{S}$ and $\sigma(W_n) \subseteq [0,1]$. From (2) we see that

  $c_\alpha\mathbf{1} - W_n - Y_n \notin G(A)$ implying that $\mathbf{1} - Y_n(c_\alpha\mathbf{1} - W_n)^{-1} \notin G(A)$.

- Since $(c_\alpha \mathbf{1} - W_n)^{-1}$ is normal for each n, we have the estimation

$$\left\|(c_\alpha \mathbf{1} - W_n)^{-1}\right\| = \rho\big((c_\alpha \mathbf{1} - W_n)^{-1}\big) \leq \frac{1}{\operatorname{dist}\big([0,1], \{c_\alpha\}\big)}$$

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- But this means that $\lim_n Y_n(c_\alpha \mathbf{1} - W_n)^{-1} = \mathbf{0}$, hence contradicting the fact that $G(A)$ is open. Therefore $c_\alpha \in [0,1]$, and thus $\phi(\alpha \mathbf{1} + x) = c_\alpha \alpha$.

Lemma 15

Let ϕ be a continuous spectrally multiplicative functional on a $C^\star$-algebra A. If $\alpha \in \mathbb{C}$ and $x \in A$ satisfies $\psi_\phi(x) = 0$, then $\phi(\alpha\mathbf{1} + x) \in \{0, \alpha\}$.

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- For each $n \in \mathbb{N}$ let $V_n := \left(\mathbf{1} + in\sqrt{|u|^2 + |v|^2}\right)^{-1}$. Again using Lemma 12, we have that

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- Let $\alpha \neq 0$. From Lemma 14, we have that $\phi(\alpha\mathbf{1} + x) = c_\alpha \alpha$, with $c_\alpha \in [0, 1]$. To obtain the result we have to show that $c_\alpha \in \{0, 1\}$: For the sake of a contradiction assume that $0 < c_\alpha < 1$. If we set $Z_n := \frac{1}{\alpha} x V_n = \frac{1}{\alpha}(u + iv)V_n$, then

$$c_\alpha = \frac{1}{\alpha}\phi(\alpha\mathbf{1} + x)\phi(V_n) \in \sigma\left(V_n + Z_n\right). \tag{3}$$

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$$c_\alpha \mathbf{1} - V_n - Z_n \notin G(A) \text{ and } c_\alpha \mathbf{1} - V_n \in G(A),$$

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Theorem 16 (Brits, Sebastian, Touré, 2022)

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- Invoking Lemma 15 again we then obtain $\phi(\alpha\mathbf{1} + x) = \alpha$ for each $\alpha \in \mathbb{C}$.
- For any value of $\psi_\phi(x)$ we use the first part of the proof to deduce that

$$\phi(x) = \phi\left(\psi_\phi(x)\mathbf{1} + [x - \psi_\phi(x)\mathbf{1}]\right) = \psi_\phi(x).$$

As a direct consequence of Theorem 16 one also has the following:

Theorem 17

Let ϕ be a continuous functional on a $C^\star$-algebra A satisfying $\phi(x)\phi(y) \in \sigma(xy)$ for all $x, y \in A$ Then. either ϕ is a character of A or $-\phi$ is a character of A.